Linear Combination

Definition: *linear combination*

A *linear combination* of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

where $a_1, \ldots, a_m \in \mathbf{F}$.

Example: In $\mathbf{F}^3$,

- $(13, -1, 7)$ is a linear combination of $(2, 1, -1), (1, -2, 4)$ because
$$5(2, 1, -1) + 3(1, -2, 4) = (13, -1, 7);$$
- $(13, -1, 6)$ is not a linear combination of $(2, 1, -1), (1, -2, 4)$ because there do not exist numbers $a_1, a_2 \in \mathbf{F}$ such that
$$a_1(2, 1, -1) + a_2(1, -2, 4) = (13, -1, 6).$$

Span

Definition: *span*

The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the *span* of $v_1, \ldots, v_m$, denoted $\operatorname{span}(v_1, \ldots, v_m)$. In other words,

$$\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbf{F}\}.$$

Example: The previous example shows that in $\mathbf{F}^3$,

- $(13, -1, 7) \in \operatorname{span}\big((2, 1, -1), (1, -2, 4)\big)$;
- $(13, -1, 6) \notin \operatorname{span}\big((2, 1, -1), (1, -2, 4)\big)$.

Span is the smallest containing subspace

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

Finite-Dimensional Vector Space

Definition: *finite-dimensional vector space*

A vector space is called *finite-dimensional* if the span of some list of vectors in it is the entire vector space.

Example: $\mathbf{F}^3$ is finite-dimensional because

$$\mathbf{F}^3 = \operatorname{span}\big((1,0,0),(0,1,0),(0,0,1)\big).$$

Definition: *infinite-dimensional vector space*

A vector space is called *infinite-dimensional* if it is not finite-dimensional.

Example: $\mathbf{F}^\infty$ is infinite-dimensional.

Linear Independence

Definition: *linearly independent*

A list $v_1, \ldots, v_m$ of vectors in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbf{F}$ that makes $a_1 v_1 + \cdots + a_m v_m$ equal 0 is $a_1 = \cdots = a_m = 0$.

Examples of linearly independent lists:

- A list v of one vector $v \in V$ is linearly independent if and only if $v \neq 0$.
- A list of two vectors in V is linearly independent if and only if neither vector is a scalar multiple of the other.
- $(1,0,0,0), (0,1,0,0), (0,0,1,0)$ is linearly independent in $\mathbf{F}^4$.
- The list $1, x, \ldots, x^m$ is linearly independent in $\mathbf{R}^{\mathbf{R}}$ for each nonnegative integer m.

Linear Dependence

Definition: ***linearly dependent***

- A list of vectors in V is called *linearly dependent* if it is not linearly independent.
- In other words, a list $v_1, \dots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \dots, a_m \in \mathbf{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

Examples of linearly dependent lists:

- $(2,3,1), (1,-1,2), (7,3,8)$ is linearly dependent in $\mathbf{F}^3$ because
$$2(2,3,1) + 3(1,-1,2) + (-1)(7,3,8) = (0,0,0).$$
- Every list of vectors in V containing the 0 vector is linearly dependent.
- If some vector in a list of vectors in V is a linear combination of the other vectors, then the list is linearly dependent.

Linear Dependence Lemma

Linear Dependence Lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such that the following hold:

- $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$;
- if the j^{th} term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$.

Example: $(2, 3, 1), (1, -1, 2), (7, 3, 8)$ is linearly dependent in $\mathbf{F}^3$.

- $(7, 3, 8) \in \operatorname{span}\big((2, 3, 1), (1, -1, 2)\big)$
- $\operatorname{span}\big((2, 3, 1), (1, -1, 2)\big) = \operatorname{span}\big((2, 3, 1), (1, -1, 2), (7, 3, 8)\big)$

Length of Linearly Independent List $\leq$ Length of Spanning List

Length of linearly independent list
$\leq$ length of spanning list

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Example: The list $(1, 2, 3), (4, 5, 8), (9, 6, 7), (-3, 2, 8)$ is not linearly independent in $\mathbf{R}^3$ because the list $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ spans $\mathbf{R}^3$.

Example: The list $(1, 2, 3, -5), (4, 5, 8, 3), (9, 6, 7, -1)$ does not span $\mathbf{R}^4$ because the list $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$ is linearly independent in $\mathbf{R}^4$.

Notation

Reminder:

$\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

V denotes a vector space over $\mathbf{F}$.

Definition: $\mathcal{P}_m(\mathbf{F})$

For m a nonnegative integer, $\mathcal{P}_m(\mathbf{F})$ denotes the set of polynomials with coefficients in $\mathbf{F}$ and degree at most m.

Examples:

- $2x^3 - 7x + 5 \in \mathcal{P}_3(\mathbf{R})$.
- $(3 + 2i)z^2 + 4iz + 9 \in \mathcal{P}_{20}(\mathbf{C})$

Definition of Basis

Definition: *basis*

A *basis* of V is a list of vectors in V that is linearly independent and spans V.

Examples of bases:

- The list $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is a basis of $\mathbf{F}^n$, called the *standard basis* of $\mathbf{F}^n$.
- The list $(1, 1, 0), (0, 0, 1)$ is a basis of $\{(x, x, y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}$.
- The list $(1, 2), (3, 5)$ is a basis of $\mathbf{F}^2$.
- The list $(1, -1, 0), (1, 0, -1)$ is a basis of
$$\{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}.$$
- The list $1, z, \ldots, z^m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

Non-examples of Bases

- The list $(1, 2, -4), (7, -5, 6)$ is linearly independent in $\mathbf{F}^3$ but is not a basis of $\mathbf{F}^3$ because it does not span $\mathbf{F}^3$.
- The list $(1, 2), (3, 5), (4, 13)$ spans $\mathbf{F}^2$ but is not a basis of $\mathbf{F}^2$ because it is not linearly independent.

Why Bases Are Useful

Basis gives unique representation as linear combination

A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in \mathbf{F}$.

Example: Let $V = \{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}$.

The list $(1, -1, 0), (1, 0, -1)$ is a basis of V.

If $(x, y, z) \in V$, then

$$(x, y, z) = -y(1, -1, 0) + (-z)(1, 0, -1).$$

Existence of Bases

Every spanning list contains a basis

Every spanning list in a vector space can be reduced to a basis of the vector space.

Basis of finite-dimensional vector space

Every finite-dimensional vector space has a basis.

Every linearly independent list extends to a basis

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Basis Length is the Same for All Bases of V

Basis length does not depend on basis

Any two bases of a vector space have the same length.

Proof Suppose B_1 and B_2 are two bases of V.

Then B_1 is linearly independent in V and B_2 spans V, so the length of B_1 is less than or equal to the length of B_2.

Interchanging the roles of B_1 and B_2, we also see that the length of B_2 is less than or equal to the length of B_1.

Thus the length of B_1 equals the length of B_2, as desired. ∎

Definition of Dimension

Definition: *dimension*, $\dim V$

- The *dimension* of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of V (if V is finite-dimensional) is denoted by $\dim V$.

Examples:

- $\dim \mathbf{F}^n = n$ because the standard basis of $\mathbf{F}^n$ has length n.
- $\dim \mathcal{P}_m(\mathbf{F}) = m + 1$ because the basis $1, z, \ldots, z^m$ of $\mathcal{P}_m(\mathbf{F})$ has length $m + 1$.

Can Check Just Linear Independence.

Linearly independent list of the right length is a basis

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Proof

Suppose $\dim V = n$ and $v_1, \ldots, v_n$ is linearly independent in V.

The list $v_1, \ldots, v_n$ can be extended to a basis of V.

However, every basis of V has length n, so in this case the extension is the trivial one, meaning that no elements are adjoined to $v_1, \ldots, v_n$.

In other words, $v_1, \ldots, v_n$ is a basis of V, as desired. ∎

Example Checking Just Linear Independence

The list $(5, 7), (4, 3)$ is a basis of $\mathbf{F}^2$.

Proof

This list of two vectors in $\mathbf{F}^2$ is linearly independent because neither vector is a scalar multiple of the other.

Note that $\mathbf{F}^2$ has dimension 2.

Thus this linearly independent list of length 2 is a basis of $\mathbf{F}^2$ (we do not need to bother checking that it spans $\mathbf{F}^2$). ∎

Can Check Just Span

Spanning list of the right length is a basis

Suppose V is finite-dimensional. Then every spanning list of vectors in V with length $\dim V$ is a basis of V.

Proof

Suppose $\dim V = n$ and $v_1, \ldots, v_n$ spans V.

The list $v_1, \ldots, v_n$ can be reduced to a basis of V.

However, every basis of V has length n, so in this case the reduction is the trivial one, meaning that no elements are deleted from $v_1, \ldots, v_n$.

In other words, $v_1, \ldots, v_n$ is a basis of V, as desired. ■

Dimension of a Sum

Dimension of a sum

If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$