Definition of Subspace

Reminder: $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

From now on, V denotes a vector space over $\mathbf{F}$.

Definition: *subspace*

A subset U of V is called a *subspace* of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Example: $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbf{F}\}$ is a subspace of $\mathbf{F}^3$.

Conditions for a Subspace

Conditions for a subspace

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

- $0 \in U$; — additive identity
- $u, w \in U$ implies $u + w \in U$; — closed under addition
- $\lambda \in \mathbf{F}$ and $u \in U$ implies $\lambda u \in U$. — closed under scalar multiplication

Examples of Subspaces

Examples of subspaces:

- If $b \in \mathbf{F}$, then
$$\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$$
is a subspace of $\mathbf{F}^4$ if and only if $b = 0$.
- The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$.
- The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$.
- The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.
- The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbf{C}^{\infty}$.
- The subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, $\mathbf{R}^2$, and all lines in $\mathbf{R}^2$ through the origin.
- The subspaces of $\mathbf{R}^3$ are precisely $\{0\}$, $\mathbf{R}^3$, all lines in $\mathbf{R}^3$ through the origin, and all planes in $\mathbf{R}^3$ through the origin.

Sums of Subspaces

Definition: *sum of subsets*

Suppose $U_1, \ldots, U_m$ are subsets of V. The *sum* of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$. More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}.$$

Sum of subspaces is the smallest containing subspace

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

Analogy: sums of subspaces $\longleftrightarrow$ unions of subsets in set theory

Direct Sums

Definition: *direct sum*

Suppose $U_1, \ldots, U_m$ are subspaces of V.

- The sum $U_1 + \cdots + U_m$ is called a *direct sum* if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j.
- If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

Example: Suppose

$$U = \{(x, y, 0) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} \quad \text{and} \quad W = \{(0, 0, z) \in \mathbf{F}^3 : z \in \mathbf{F}\}.$$

Then $\mathbf{F}^3 = U \oplus W$.

Direct Sums

Condition for a direct sum

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0.

Direct sum of two subspaces

Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.