Real Numbers

Notation: $\mathbf{R}$

$\mathbf{R}$ denotes the field of real numbers.

Some nonconstant polynomials with real coefficients have no real zeros. Example: the equation

$$x^2 + 1 = 0$$

has no real solutions.

Thus we invent a solution, called i, with the property that $i^2 = -1$.

Complex Numbers

Definition: *complex numbers*

- A *complex number* is an ordered pair (a, b), where $a, b \in \mathbf{R}$, but we will write this as $a + bi$.
- The set of all complex numbers is denoted by $\mathbf{C}$:

$$\mathbf{C} = \{a + bi : a, b \in \mathbf{R}\}.$$

- *Addition and multiplication* on $\mathbf{C}$ are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

If $a \in \mathbf{R}$, we identify $a + 0i$ with the real number a. Thus we think of $\mathbf{R}$ as a subset of $\mathbf{C}$. We also usually write $0 + bi$ as just bi, and we usually write $0 + 1i$ as just i. The definition of multiplication shows that $i^2 = -1$.

Properties of Complex Arithmetic

- **commutativity**
  $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbf{C}$;
- **associativity**
  $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$;
- **identities**
  $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbf{C}$;
- **additive inverse**
  for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$;
- **multiplicative inverse**
  for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$;
- **distributive property**
  $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

Notation: $\mathbf{F}$

$\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

Elements of $\mathbf{F}$ are sometimes called *scalars*, which is just a fancy word for numbers.

$\mathbf{R}^2$ and $\mathbf{R}^3$

Definition: $\mathbf{R}^2$ *and* $\mathbf{R}^3$

- The set $\mathbf{R}^2$, which you can think of as a plane, is the set of all ordered pairs of real numbers:

$$\mathbf{R}^2 = \{(x, y) : x, y \in \mathbf{R}\}.$$

- The set $\mathbf{R}^3$, which you can think of as ordinary space, is the set of all ordered triples of real numbers:

$$\mathbf{R}^3 = \{(x, y, z) : x, y, z \in \mathbf{R}\}.$$

Lists

Fix a positive integer n.

Definition: *list, length*

A *list* of *length* n is an ordered collection of n numbers separated by commas and surrounded by parentheses.

Example: $(7, 3)$ is a list of length 2. Thus $(7, 3) \in \mathbf{R}^2$.

Example: $(5, 9, -2)$ is a list of length 3. Thus $(5, 9, -2) \in \mathbf{R}^3$.

A list of length n looks like this:

$$(x_1, \ldots, x_n).$$

Two lists are equal if and only if they have the same length and the same elements in the same order.

Definition: $\mathbf{F}^n$

$\mathbf{F}^n$ is the set of all lists of length n of elements of $\mathbf{F}$:

$$\mathbf{F}^n = \{(x_1, \dots, x_n) : x_j \in \mathbf{F} \text{ for } j = 1, \dots, n\}.$$

Example: $\mathbf{R}^4$ is the set of all lists of four real numbers:

$$\mathbf{R}^4 = \{(x, y, z, w) : x, y, z, w \in \mathbf{R}\}.$$

Example: $\mathbf{C}^4$ is the set of all lists of four complex numbers:

$$\mathbf{C}^4 = \{(z_1, z_2, z_3, z_4) : z_1, z_2, z_3, z_4 \in \mathbf{C}\}.$$

Elements of $\mathbf{F}^n$ are often called *points* or *vectors*.

Addition and Scalar Multiplication

Definition: *addition in $\mathbf{F}^n$*

Addition in $\mathbf{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n).$$

Example: $(3, 4, -1, -2) + (5, 0, 6, -7) = (8, 4, 5, -9)$

Definition: *scalar multiplication in $\mathbf{F}^n$*

The *product* of a number $\lambda \in \mathbf{F}$ and a vector in $\mathbf{F}^n$ is computed by multiplying each coordinate of the vector by λ:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n).$$

Example: $i(i, 2, 5 + i, -3i) = (-1, 2i, -1 + 5i, 3)$

Single Letters Can Denote Elements of $\mathbf{F}^n$

Using a single letter to denote elements of $\mathbf{F}^n$ is often efficient.

Commutativity of addition in $\mathbf{F}^n$

If $x, y \in \mathbf{F}^n$, then $x + y = y + x$.

Definition: 0

Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \ldots, 0).$$

Multiplication by 0

If $x \in \mathbf{F}^n$, then $0x = 0$.

The 0 above on the left is the number 0.
The 0 above on the right is the vector 0 in $\mathbf{F}^n$.

Motivation for Vector Space Definition

Reminder: $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

The motivation for the definition of a vector space comes from properties of addition and scalar multiplication in $\mathbf{F}^n$:

- Addition is commutative, associative, and has an identity.
- Every element has an additive inverse.
- Scalar multiplication is associative.
- Scalar multiplication by 1 acts as expected.
- Addition and scalar multiplication are connected by distributive properties.

Addition and Scalar Multiplication

Definition: ***addition, scalar multiplication***

- An *addition* on a set V is a function that assigns an element $u + w \in V$ to each pair of elements $u, w \in V$.
- A *scalar multiplication* on a set V is a function that assigns an element $\lambda u \in V$ to each $\lambda \in \mathbf{F}$ and each $u \in V$.

Example: Suppose V is the set of real-valued functions on the interval $[0, 1]$. For $f, g \in V$ and $\lambda \in \mathbf{R}$, define $f + g$ and λf by

$$(f + g)(x) = f(x) + g(x)$$

and

$$(\lambda f)(x) = \lambda f(x).$$

Thus $f + g \in V$ and $\lambda f \in V$.

Definition of Vector Space

Definition: *vector space*

A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

- commutativity: $u + w = w + u$ for all $u, w \in V$;
- associativity: $(u + v) + w = u + (v + w)$ and $(ab)u = a(bu)$ for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$;
- additive identity: there exists $0 \in V$ such that $u + 0 = u$ for all $u \in V$;
- additive inverse: for every $u \in V$, there exists $w \in V$ such that $u + w = 0$;
- multiplicative identity: $1u = u$ for all $u \in V$;
- distributive properties: $a(u + w) = au + aw$ and $(a + b)u = au + bu$ for all $a, b \in \mathbf{F}$ and all $u, w \in V$.

Examples

- $\mathbf{F}^n$ with the usual operations of addition and scalar multiplication is a vector space.
- $\mathbf{F}^\infty$ is defined to be the set of all sequences of elements of $\mathbf{F}$:

  $$\mathbf{F}^\infty = \{(x_1, x_2, \dots) : x_j \in \mathbf{F} \text{ for } j = 1, 2, \dots\}.$$

  Addition and scalar multiplication on $\mathbf{F}^\infty$ are defined as expected:

  $$(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1 + y_1, x_2 + y_2, \dots),$$
  $$\lambda(x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \dots).$$

  With these definitions, $\mathbf{F}^\infty$ becomes a vector space.

Another example

If S is a set, then $\mathbf{F}^S$ denotes the set of functions from S to $\mathbf{F}$.

For $f, g \in \mathbf{F}^S$, the *sum* $f + g \in \mathbf{F}^S$ is the function defined by

$$(f+g)(x) = f(x) + g(x)$$

for all $x \in S$.

For $\lambda \in \mathbf{F}$ and $f \in \mathbf{F}^S$, the *product* $\lambda f \in \mathbf{F}^S$ is the function defined by

$$(\lambda f)(x) = \lambda f(x)$$

for all $x \in S$.

With these definitions of addition and scalar multiplication, $\mathbf{F}^S$ becomes a vector space.

Our First Theorem

The number 0 times a vector

If V is a vector space, then $0u = 0$ for every $u \in V$.

Proof For $u \in V$, we have

$$\begin{aligned} 0u &= (0+0)u \\ &= 0u + 0u. \end{aligned}$$

Adding the additive inverse of $0u$, denoted $-0u$, to both sides of the equation above gives

$$0u + (-0u) = 0u + 0u + (-0u),$$

which can be rewritten as

$$0 = 0u,$$

as desired. ∎

Why Abstraction?

Advantages of the abstract approach:

- Can apply in multiple new situations.
- Stripping away inessential properties leads to greater understanding.

If V is a vector space, it would be incorrect to prove that $0u = 0$ for $u \in V$ by writing: Let $u = (x_1, \ldots, x_n)$, thus

$$\begin{aligned} 0u &= (0x_1, \ldots, 0x_n) \\ &= (0, \ldots, 0) \\ &= 0. \end{aligned}$$

An element of V is not necessarily of the form $(x_1, \ldots, x_n)$.